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COMMENT

Comment on the q-analogues of the harmonic oscillator

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Abstract. The Heisenberg relation for the q-analogues of the quantum harmonic oscillator (introduced independently by Macfarlane and Biedenharn for the quantum group $SU(2)_q$) is derived by a method analogous to that used by Schwinger for the SU(2) case. I also speculate on the possible use of the quantum group as a generalisation of quantum mechanics.

Recently there has been a great deal of interest in the study of the quantum groups† in the context of exactly soluble statistical models, integrable systems in field theory, non-commutative geometry and other fields. Of particular interest here is the development by Macfarlane [2] and independently by Biedenharn [3] of a new realisation of the quantum group $SU(2)_q$ in terms of the q-analogues of the quantum harmonic oscillator. If the important role played by the harmonic oscillator system in physics is any guide, it is quite possible that the q-oscillator plays a useful role in the study of $SU(2)_q$. In this Comment we will discuss a method (analogous to that used by Schwinger [4] in his development of the quantum theory of angular momentum) to derive the Heisenberg relation for the q-oscillator in its symmetrical form. Its consistency with the unsymmetrical form found in [2] and independently postulated in [3] will be shown, and other forms will be given. We will also discuss briefly the case of fermionic q-oscillators.

The quantum group $SU(2)_q$ is a q-deformation of the Lie algebra of SU(2). It is generated by three Hermitian operators J_1, J_2, J_3 or the more convenient combination J_+, J_-, J_3 that obey the commutation relations

$$[J_3, J_\pm] = \pm J_\pm \tag{1}$$

$$[J_+, J_-] = [2J_3] \tag{2}$$

where $J_\pm = J_1 \pm \mathrm{i}J_2$ and we have introduced the notation [ ] defined in terms of a parameter q (taken to be real for simplicity)

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}} \tag{3}$$

or, in terms of $\gamma = \log q$,

$$[x] = \frac{\mathrm{e}^{\gamma x} - \mathrm{e}^{-\gamma x}}{\mathrm{e}^{\gamma} - \mathrm{e}^{-\gamma}} = \frac{\sinh \gamma x}{\sinh \gamma}. \tag{4}$$

† For a review see, e.g., [1] and references contained therein.

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We note the following properties of $[x]$. (i) In the limit $q \to 1$, i.e. $\gamma \to 0$, $[x] \to x$; (ii) $[-x] = -[x]$; (iii) $[1] = 1$ and $[0] = 0$†. More importantly, [ ] is invariant under what we will call the duality transformation $q \leftrightarrow q^{-1}$, i.e. $\gamma \leftrightarrow -\gamma$.

We start with Jimbo's representation [5] of (1) and (2), which acts in a Hilbert space with basis $|jm\rangle$ according to

$$J_3|jm\rangle = m|jm\rangle \tag{5}$$

$$J_\pm|jm\rangle = ([j \mp m][j \pm m + 1])^{1/2}|jm \pm 1\rangle \tag{6}$$

where $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ and $-j \leqslant m \leqslant j$. In the limit $\gamma \to 0$, (5) and (6) reduce to the familiar result of the quantum theory of angular momentum. Equation (6) can be used successively to give

$$J_- J_+|jm\rangle = [j - m][j + m + 1]|jm\rangle \tag{7}$$

$$J_+ J_-|jm\rangle = [j + m][j - m + 1]|jm\rangle. \tag{8}$$

Following Schwinger [4], we consider the $j \to \infty$ limit of (7) and (8) in the following way. For the upper end portion of the spectrum $m \leqslant j$, the difference between j and m is finite, i.e.

$$j \to \infty \quad m \to \infty \qquad j - m = n_1 = 0, 1, 2, 3, \ldots. \tag{9}$$

In this large-j and large-m limit, one can use (7) and (8) respectively to yield (omitting the state $|jm\rangle$ on both sides)

$$\frac{J_-}{\sqrt{[2j]}} \frac{J_+}{\sqrt{[2j]}} = [j - m] \frac{[j + m + 1]}{[2j]} \to [n_1] \tag{10}$$

$$\frac{J_+}{\sqrt{[2j]}} \frac{J_-}{\sqrt{[2j]}} = \frac{[j + m]}{[2j]} [j - m + 1] \to [n_1 + 1]. \tag{11}$$

Defining

$$a_1 = \frac{J_+}{\sqrt{[2j]}} \tag{12}$$

$$a_1^\dagger = \frac{J_-}{\sqrt{[2j]}} \tag{13}$$

we can rewrite (10) and (11) respectively as

$$a_1^\dagger a_1 = [N_1] \tag{14}$$

$$a_1 a_1^\dagger = [N_1 + 1] \tag{15}$$

where we have recalled that J_+ and J_-, and accordingly a_1 and $a_1^\dagger$, are operators and have introduced the number operator N_1 which takes on values $n_1 = 0, 1, 2, \ldots$. The difference between (15) and (14) yields the commutator of a and $a^\dagger$ according to

$$[a_1, a_1^\dagger] = [N_1 + 1] - [N_1] = \frac{\cosh[(\gamma/2)(2N_1 + 1)]}{\cosh(\gamma/2)}. \tag{16}$$

† Thus both the $SU(2)_q$ and SU(2) give the same algebras for spin 0 ($J_3 = 0$) and for spin-$\frac{1}{2}$ ($2J_3 = \pm 1$) systems, but different algebras for higher-spin systems.

For the sake of completeness, we remark that one can now construct a Hilbert space with basis $|n_1\rangle$ such that

$$N_1|n_1\rangle = n_1|n_1\rangle \qquad |n_1\rangle = \frac{(a_1^\dagger)^{n_1}}{([n_1]!)^{1/2}}|0\rangle \qquad a_1|0\rangle = 0 \tag{17}$$

with

$$a_1^\dagger|n_1\rangle = [n_1+1]^{1/2}|n_1+1\rangle \qquad a_1|n_1\rangle = [n_1]^{1/2}|n_1-1\rangle \tag{18}$$

where

$$[n_1]! \equiv [n_1][n_1-1]\ldots[2][1]. \tag{19}$$

In the limit $\gamma \to 0$ (14)-(19) reproduce the familiar harmonic oscillator results, in particular

$$[a, a^\dagger] = 1 \qquad \gamma = 0 \tag{20}$$

the old Heisenberg relation for the quantised harmonic oscillator.

We can repeat the above procedure for the large-$|m|$ portion of the lower end of the $j \to \infty$ spectrum, with $j+m = n_2 = 0, 1, 2, 3, \ldots$. To describe it one introduces another (independent) harmonic oscillator (labelled by subscript 2) obeying the same algebra (14)-(19) as the first oscillator (with subscript 1). The whole $SU(2)_q$ spectrum can now be described by two commuting oscillators so that

$$|jm\rangle = \frac{(a_1^\dagger)^{j-m}}{([j-m]!)^{1/2}} \frac{(a_2^\dagger)^{j+m}}{([j+m]!)^{1/2}}|0\rangle \tag{21}$$

and the original $SU(2)_q$ algebra (1) and (2) can be satisfied with the following identification:

$$J_- = a_1^\dagger a_2 \qquad J_+ = a_2^\dagger a_1 \qquad 2J_3 = N_2 - N_1. \tag{22}$$

(For the last identification, it is helpful to note that $[N_1][N_2+1]-[N_1+1][N_2] = [N_1-N_2]$.) Thus we have a new realisation [2, 3] for $SU(2)_q$ analogous to that of Schwinger, Bargmann and Jordan† for the algebra of quantum angular momentum.

It remains for us to show that the Heisenberg relation for the q-oscillator given by (16) is consistent with that found in [2] and [3]. The Heisenberg relation proposed in [2] and [3] has the following unsymmetrical form:

$$aa^\dagger - q^{-1}a^\dagger a = q^N \tag{23}$$

which we rewrite as

$$q^{1/2}aa^\dagger - q^{-1/2}a^\dagger a = q^{N+1/2}. \tag{24a}$$

But we can use the fact that [ ] is invariant under the duality transformation $q \leftrightarrow q^{-1}$ to cast (24a) in the form

$$q^{-1/2}aa^\dagger - q^{1/2}a^\dagger a = q^{-(N+1/2)} \tag{24b}$$

where we have used (14) and (15). The consistency between (16) and (24) is now obvious.

† See [6].

Actually (16) is only one particular form of the Heisenberg relation (albeit the most natural one). Using (14) and (15) one can express the 'Heisenberg relation' in many different ways, e.g. (23) (the unsymmetrical one proposed in [2] and [3]), its dual form

$$aa^{\dagger} - qa^{\dagger}a = q^{-N} \tag{25}$$

and†

$$aa^{\dagger} + q^{2N+1}a^{\dagger}a = q^{N}[2N+1]. \tag{26}$$

One can also discuss the fermionic q-oscillators. The number operator N, for this case, can take on values $n = 0$, 1 only. The Hilbert space with basis $|n\rangle$ is constructed such that

$$\begin{aligned} &N|n\rangle = n|n\rangle \qquad |1\rangle = b^{\dagger}|0\rangle \\ &b|0\rangle = 0 = b^{\dagger}|1\rangle \\ &\{b, b\} = 0 = \{b^{\dagger}, b^{\dagger}\} \end{aligned} \tag{17'}$$

where we have used { , } to denote the anticommutator and b, $b^{\dagger}$ are respectively the annihilation and creation operators for the fermionic q-oscillators. The fermionic analogues of (14) and (15) are respectively

$$b^{\dagger}b = [N] \rightarrow N \tag{14'}$$

$$bb^{\dagger} = [1-N] \rightarrow 1-N \tag{15'}$$

from which follows the Heisenberg relation

$$\{b, b^{\dagger}\} = \frac{\cosh[(\gamma/2)(2N-1)]}{\cosh(\gamma/2)} \rightarrow 1 \tag{16'}$$

where we have made use of the fact that N can only take on values 1 or 0. We can also express the 'Heisenberg relation' in the following forms (which are the fermionic analogues of (23) and (26) respectively):

$$bb^{\dagger} + qb^{\dagger}b = q^{N} \tag{23'}$$

$$bb^{\dagger} - q^{2N-1}b^{\dagger}b = q^{N}[1-2N]. \tag{26'}$$

We conclude with some speculations on the possible use of the quantum group as a generalisation of quantum mechanics. The $SU(2)_q$ algebra (1) and (2) is obviously a simple generalisation of the usual quantum angular momentum algebra. The question is: what role does the parameter γ play? At least three possible roles come to mind. (i) Quantum mechanics makes use of flat Hilbert spaces; in the generalised quantum mechanics, γ may play the role of a small curvature of the Hilbert spaces. (ii) Quantum mechanical descriptions of a quantum system under study ignores the quantum effects of the system on the observer; in the generalised quantum mechanics, γ may parametrise the back reactions of the system on the observer and the resulting additional effects of the observer on the system and so on. (Note that even the form of [ ] suggests an infinite series.) (iii) The $SU(2)_q$ algebra (1) and (2) treats the third direction (along which quantum measurements are made) differently from the first and second directions.

† Although normally one does not regard (26) (or similar expressions) as a form of Heisenberg relation.

If the operators J_i are taken to be the generators of rotations†, then, according to the $SU(2)_q$ algebra, rotational symmetry is broken and γ is a measure of the symmetry breaking‡§. Finally we note that if indeed $\gamma \neq 0$ it is also conceivable that γ is given by a ratio of quantities associated with the system under consideration and some dimensionful physical constants; one of them may even be a new physical constant! Although $\gamma \approx 0$ at low energies, it may take on significantly different values at high energies (for example, in the early universe?) These are all interesting speculations. But independent of them, the physical relevance of the quantum group is beyond doubt. The q-oscillators may yet play a useful role in physics.

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† Related questions include how to define the orbital angular momentum in terms of coordinates $\boldsymbol{r}$ and linear momenta $\boldsymbol{p}$ such that (1) and (2) are satisfied. Furthermore, do $\boldsymbol{r}$ and $\boldsymbol{p}$ appropriately obey the canonical commutation relations? What is the classical (Poisson bracket) analogue of (1) and (2)? Or is it possible that the orbital angular momenta obey the standard SU(2) algebra while the intrinsic spins obey the new $SU(2)_q$ algebra?

‡ In statistical mechanical models γ is a measure of anisotropy.

§ The idea of spontaneous breaking of rotational symmetries may be related to the fact that, of all possible directions, one makes a quantum measurement along one particular direction.